Cohomology of Deligne-Lusztig varieties for groups of type A

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Abstract

We study the cohomology of parabolic Deligne-Lusztig varieties associated to unipotent blocks of $\mathrm{GL}_n(q)$. We show that the geometric version of Broué's conjecture over $\overline{\mathbb{Q}}_\ell$, together with Craven's formula, holds for any unipotent block whenever it holds for the principal Φ_1-block.

Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$ with an $\mathbb{F}_q$-structure associated to a Frobenius endomorphism F. Let ℓ be a prime number different from p and b be a unipotent ℓ-block of $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$. When ℓ is large, the defect group of b is abelian, and the geometric version of Broué's conjecture predicts that the cohomology of some Deligne-Lusztig variety should induce a derived equivalence between b and its Brauer correspondent [2].

When the centraliser of the defect group of b is a torus, then in [4] Broué and Michel identified which specific class of Deligne-Lusztig varieties should be considered. They correspond to good d-regular elements or equivalently to d-roots of $\boldsymbol{\pi} = \mathbf{w}_0^2$ in the Braid monoid. In a recent work [9], Digne and Michel introduced the notion of *d-periodic element* to generalise this to the parabolic setting. If b is a unipotent Φ_d-block, then it is to be expected that there exists a d-periodic element $(\mathbf{I}, \mathbf{w})$ such that the corresponding parabolic Deligne-Lusztig variety $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{w}F)$ is a good candidate for inducing the derived equivalence predicted by Broué's conjecture. Furthermore, Chuang and Rouquier conjectured in [6] that this equivalence is perverse, with a perversity function that has recently been conjectured by Craven in [7]. Surprisingly, it can be expressed by a function C_d depending only on the generic degrees of the corresponding characters.

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If we restrict our attention to the characteristic zero then we obtain a conjectural explicit description of the unipotent part of the cohomology of $\widetilde{\mathrm{X}}(\mathbf{I},\mathbf{w}F)$. The fundamental property that we derive from Broué's conjecture is that the cohomology groups of $\mathrm{X}(\mathbf{I},\mathbf{w}F)$ are mutually disjoint. More precisely, it can be formulated as follows:

Conjecture 1. *Any d-cuspidal pair is conjugate to a pair $(\mathbf{L}_I^{\mathbf{w}F},\chi)$ where $(\mathbf{I},\mathbf{w})$ is a d-periodic element. Moreover, if $\mathcal{F}_\chi$ is the corresponding $\overline{\mathbb{Q}}_\ell$-local system on $\mathrm{X}(\mathbf{I},\mathbf{w}F)$, then $(\mathbf{I},\mathbf{w})$ can be chosen such that*

(i) *The $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-modules $\mathrm{H}^i\big(\mathrm{X}(\mathbf{I},\mathbf{w}F),\mathcal{F}_\chi\big)$ are mutually disjoint.*

(ii) *If ρ is a irreducible unipotent constituent of $\mathrm{H}^i\big(\mathrm{X}(\mathbf{I},\mathbf{w}F),\mathcal{F}_\chi\big)$ then ρ lies in the Φ_d-block associated to χ and $i = C_d(\deg\rho/\deg\chi)$.*

In addition, the endomorphism algebra $\mathrm{End}_{\mathbf{G}^F}\big(\mathrm{H}^\bullet(\mathrm{X}(\mathbf{I},\mathbf{w}F),\mathcal{F}_\chi)\big)$ should be endowed with a natural structure of d-cyclotomic Hecke algebra. Let us note the following important consequence of this property: the eigenvalue of any sufficiently divisible power F^m of F on ρ should be $q^{m(a_\rho+A_\rho-a_\chi-A_\chi)/d}$.

The choice of a specific d-periodic element in this conjecture is not very relevant: it is conjectured that any other d-periodic element $(\mathbf{I},\mathbf{w}')$ can be obtained from $(\mathbf{I},\mathbf{w}')$ by cyclic shifts, so that the cohomology of the corresponding varieties are isomorphic. This has already been proven when $\mathbf{I}=\varnothing$ and F acts trivially on W (see [9, Remark 7.4]).

When $d=1$, the unipotent blocks correspond to the usual Harish-Chandra series. In particular, when $(\mathbf{G},F)$ has type A, there is a unique unipotent block and it contains all the unipotent characters. The purpose of this paper is to show that from the cohomology of $\mathrm{X}(\boldsymbol{\pi})$ one can actually deduce all the other interesting cases (see Corollary 3.2):

Theorem. *For groups of type A, Conjecture 1 holds whenever it holds for $d=1$, that is for $\mathrm{X}(\boldsymbol{\pi})$.*

Let us emphasize that Conjecture 1 is known to be true only in a very small number of cases, namely when $d=h$ is the Coxeter number by Lusztig [12], for groups of rank 2 by Digne, Michel and Rouquier [10] and when $d=n$ for A_n and $d=4$ for D_4 by Digne and Michel [8]. Therefore this theorem represents a very important step towards a proof of the geometric version of Broué's conjecture.

Even though this result depends on the conjectural description of $\mathrm{X}(\boldsymbol{\pi})$, one can give an effective proof of Conjecture 1 for principal Φ_d-blocks when $d > (n+1)/2$. In that case the defect group is cyclic, and the modular representation theory of the block is fully understood. We will address this problem in a subsequent paper, where we will compute the cohomology of $\overline{\mathbb{Z}}_\ell$ of the corresponding Deligne-Lusztig variety.

To give a flavour of the proof of the main theorem, recall that for groups of type A, we have by [3] a combinatorial description of the Deligne-Lusztig

induction associated to $\widetilde{\mathrm{X}}(\mathbf{I},\mathbf{w}F)$. In terms of partitions, it corresponds to adding a certain number of d-hooks. By transitivity, one can decompose $\widetilde{\mathrm{X}}(\mathbf{I},\mathbf{w}F)$ by means of simpler varieties $\widetilde{\mathrm{X}}_{n,d}$, each of which corresponds to adding a single d-hook to a partition. Now, using the methods developed in [11] one can compute the cohomology of (some quotient of) $\widetilde{\mathrm{X}}_{n,d}$ in terms of $\widetilde{\mathrm{X}}_{n-1,d}$ and $\widetilde{\mathrm{X}}_{n-1,d-1}$ (see Theorem 2.1), providing an inductive argument to tackle Conjecture 1.

Note finally that Theorem 2.1 can be generalised to many other situations in type B, C and D. However, two main problems arise: firstly, the limit case is either $\mathrm{X}(w_0)$ or $\mathrm{X}(\boldsymbol{\pi})$ and does not contain all the unipotent characters. Secondly, the methods in [11] work obviously for non-cuspidal unipotent characters only. We can obtain partial results on the principal series in that situation, which we believe are too coarse to be mentioned in this paper.

1 Parabolic varieties in type A

Throughout this paper, $\mathbf{G}$ will denote any connected reductive algebraic group of type A_n over $\mathbb{F}=\overline{\mathbb{F}}_p$. We will consider a Frobenius endomorphism $F:\mathbf{G}\longrightarrow\mathbf{G}$ defining a standard $\mathbb{F}_q$-structure on $\mathbf{G}$. Since we will be interested in unipotent characters only, we will not make any specific choice for $(\mathbf{G},F)$ in its isogeny class. If $\mathbf{H}$ is any F-stable subgroup of $\mathbf{G}$ we will denote by $H=\mathbf{H}^F$ the associated finite group.

The Weyl group W of $\mathbf{G}$ is the symmetric group $\mathfrak{S}_n$ and its Braid monoid B^+ is the usual Artin monoid. It is generated by a set $\mathbf{S}=\{\mathbf{s}_1,\ldots,\mathbf{s}_n\}$ corresponding to simple reflections $s_1,\ldots,s_n$ of W. Following [9], we define for $1\le d\le n+1$

$$\mathbf{v}_d=\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_{n-\lfloor\frac{d}{2}\rfloor}\mathbf{s}_n\mathbf{s}_{n-1}\cdots\mathbf{s}_{\lfloor\frac{d+1}{2}\rfloor}$$

and
$$\mathbf{J}_d=\left\{\mathbf{s}_i\mid\lfloor\tfrac{d+1}{2}\rfloor+1\le i\le n-\lfloor\tfrac{d}{2}\rfloor\right\}\subset\mathbf{S}.$$

We are interested in computing the cohomology of the variety

$$\mathrm{X}_{n,d}=\mathrm{X}(\mathbf{J}_d,\mathbf{v}_dF)$$

with coefficients in any unipotent local system. Note that for $d>1$, the element $\mathbf{v}_d$ is reduced so that we can work with the variety $\mathrm{X}(J_d,v_dF)$. By [9, Lemma 11.7 and 11.8], the pair $(\mathbf{J}_d,\mathbf{v}_d)$ is *d-periodic* so that it makes sense to study the cohomology of $\mathrm{X}_{n,d}$. Recall from [9] that a d-periodic element is any pair $(\mathbf{I},\mathbf{b})$ with $\mathbf{I}\subset\mathbf{S}$ and $\mathbf{b}\in B^+$ such that $\mathbf{b}F(\mathbf{b})\cdots F^{d-1}(\mathbf{b})=\boldsymbol{\pi}/\boldsymbol{\pi}_I$ where $\boldsymbol{\pi}=\mathbf{w}_0^2$ is the generator of the pure Braid group. It has been shown in [9] that this forces $\mathbf{b}F$ to normalise $\mathbf{I}$. Note that when $d\le(n+1)/2$, $\mathbf{v}_d$ is not *maximal* in the sense that it is not extendable to a dth root of $\boldsymbol{\pi}/\boldsymbol{\pi}_I$ for a proper subset $\mathbf{I}$ of $\mathbf{J}_d$. However, it can still be used to associate to any unipotent block a "good" parabolic Deligne-Lusztig variety. Before making any precise statement, we shall briefly recall the combinatorial objects that we will use.

1.1. **Φ_d-blocks of G.** The unipotent characters of G are labeled by the partitions of $n+1$. If λ is such a partition, we will denote by χ_λ the corresponding character,

with the convention that $\chi_{(1,1,\ldots,1)} = \mathrm{St}_G$ is the Steinberg character of G. We shall also fix a representation V_λ over $\overline{\mathbb{Q}}_\ell$ of character χ_λ. For $1 \leq d \leq n+1$, the pair $(\mathbf{L}_{J_d}, v_d F)$ represents a d-Levi subgroup of $\mathbf{G}$. From [3], we know how to express the d-Harish-Chandra induction in terms of combinatorics of partitions. To fix the notation, let μ be a partition of $n+1-d$ and $X = \{x_1 < x_2 < \cdots < x_s\}$ be a β-set associated to μ. We may and we will assume that X is big enough, so that it contains $\{0,1,\cdots,d-1\}$. Let X' be the subset of X defined by $X' = \{x \in X \mid x+d \notin X\}$. It represents the possible d-hooks that can be added to μ. For $x \in X'$ we will denote by $\mu * x$ the partition of $n+1$ which has $(X \smallsetminus \{x\}) \cup \{x+d\}$ as a β-set.

We fix an F-stable Tits homomorphism $t : B^+ \longrightarrow N_{\mathbf{G}}(\mathbf{T})$. By [9] the variety $\mathrm{X}_{n,d}$ has an étale covering $\widetilde{\mathrm{X}}_{n,d} = \widetilde{\mathrm{X}}(\mathbf{J}_d, \mathbf{v_d}F)$ with Galois group $\mathbf{L}_{J_d}^{t(\mathbf{v_d})F}$. Since $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$ is a split group of type A_{n-d}, the partition μ defines a unipotent local system $\mathcal{F}_\mu$ on $\mathrm{X}_{n,d}$ such that $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d}, \mathcal{F}_\mu)$ and $\mathrm{H}_c^\bullet(\widetilde{\mathrm{X}}_{n,d}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$ are isomorphic. Then we deduce from [3, Section 3.4] that there exist signs $\varepsilon_x = \pm 1$ such that the d-Harish-Chandra induction of χ_μ is given by

$$\mathrm{R}_{\mathbf{L}_{J_d}}^{\mathbf{G}}(\chi_\mu) = \sum (-1)^i \mathrm{H}_c^i(\mathrm{X}_{n,d}, \mathcal{F}_\mu) = \sum_{x \in X'} \varepsilon_x \chi_{\mu * x}.$$

In particular, the d-Harish-Chandra restriction of $\chi_\lambda \in \mathrm{Irr}\, G$ is non-zero until we reach the d-core ν of λ, which corresponds to a d-cuspidal character χ_ν. The unipotent characters in the Φ_d-block of G containing χ_λ are all the characters that can be obtained by successive d-inductions from χ_ν. They correspond to partitions of $n+1$ that have ν as a d-core.

1.2. **A parabolic variety associated to a Φ_d-block.** The cohomology of the variety $\mathrm{X}_{n,d}$ induces to a minimal d-induction since there is no d-split Levi between $(\mathbf{L}_{J_d}, t(\mathbf{v_d})F)$ and $(\mathbf{G}, F)$. By transitivity, one can form a Deligne-Lusztig variety $\mathrm{X}(\mathbf{I}, \mathbf{w})$ associated to the d-cuspidal character χ_ν. Let $n+1-ad$ be the size of ν and consider for $i = 1, \ldots, a$ the pairs $(\mathbf{J}_d^{(i)}, \mathbf{v}_d^{(i)})$ where $(\mathbf{J}_d^{(0)}, \mathbf{v}_d^{(0)}) = (\mathbf{J}_d, \mathbf{v}_d)$ and $(\mathbf{J}_d^{(i+1)}, \mathbf{v}_d^{(i+1)})$ is the analogue of the pair $(\mathbf{J}_d, \mathbf{v}_d)$ for the split group $(\mathbf{L}_{J_d^{(i)}}, t(\mathbf{v}_d^{(i)} \cdots \mathbf{v}_d^{(1)})F)$ of type A_{n-id}. Then one can readily check that the pair $(\mathbf{I}, \mathbf{w}) = (\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} \cdots \mathbf{v}_d^{(1)})$ is d-periodic.

By [9, Proposition 8.26] the associated Deligne-Lusztig variety $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{w}F)$ is isomorphic to the following almalgamated product

$$\widetilde{\mathrm{X}}(\mathbf{J}_d^{(1)}, \mathbf{v}_d^{(1)}F) \times_{\mathbf{L}_{J_d^{(1)}}^{t(\mathbf{v}_d^{(1)})F}} \cdots \times_{\mathbf{L}_{J_d^{(a-1)}}^{t(\mathbf{v}_d^{(a-1)} \cdots \mathbf{v}_d^{(1)})F}} \widetilde{\mathrm{X}}_{\mathbf{L}_{\mathbf{J}_d^{(a-1)}}}(\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} t(\mathbf{v}_d^{(a-1)} \cdots \mathbf{v}_d^{(1)})F).$$

Now each variety in this decomposition corresponds to a variety $\widetilde{\mathrm{X}}_{n-id,d}$ for some $i = 0, \ldots, a-1$. Since the cohomology of the latter with coefficients in a unipotent local system depends only on the isogeny class of the group (here, the split type A_{n-id}) we obtain

$$\mathrm{R}\Gamma_c\big(\mathrm{X}(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\nu\big) \simeq \mathrm{R}\Gamma_c(\widetilde{\mathrm{X}}_{n,d}, \overline{\mathbb{Q}}_\ell) \otimes_{A_{n-d}} \cdots \otimes_{A_{n-(a-1)d}} \mathrm{R}\Gamma_c(\mathrm{X}_{n-(a-1)d,d}, \mathcal{F}_\nu). \quad (1.1)$$

Consequently, if we believe that the vanishing property in Conjecture 1 holds for the cohomology of $\mathrm{X}(\mathbf{I},\mathbf{w})$, then for any partition μ of $n+1-d$, the graded G-module $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)$ should be multiplicity-free.

1.3. **Craven's formula in type A.** Conjecturally, the unipotent character χ_λ is a constituent of only one cohomology group of $\mathrm{R}\Gamma\big(\mathrm{X}(\mathbf{I},\mathbf{w}F),\mathcal{F}_\nu\big)$. Craven proposed in [7] a formula which gives the degree of this cohomology group in terms of d and the generic degree of χ_λ and χ_μ. More precisely, he considers a function C_d on some set of enhanced cyclotomic polynomials and conjectured that

$$\langle \chi_\lambda\,;\mathrm{H}^i(\mathrm{X}(\mathbf{I},\mathbf{w}),\mathcal{F}_\nu)\rangle_G \neq 0 \iff i = C_d(\deg\chi_\lambda)-C_d(\deg\chi_\nu). \tag{1.2}$$

Let us recall the definition of C_d: assume that $P\subset\mathbb{Q}[x]$ is a polynomial such that the non-zero roots $z_1,\ldots,z_m$ (written with multiplicity) of P are all roots of unity. Let us denote by $d^\circ(P)$ the degree of P and by $v(P)$ its valuation, that is the degree of $x^{d^\circ(P)}P(x^{-1})$. Then Craven's function C_d is defined by

$$C_d(P)=\frac{1}{d}\big(d^\circ(P)+v(P)\big)+\#\big\{i=1,\ldots,m\,|\,\mathrm{Arg}\,z_i<2\pi/d\big\}-\frac{1}{2}\#\big\{i=1,\ldots,m\,|\,z_i=1\big\}.$$

Here, the argument $\mathrm{Arg}\,z$ of a non-zero complex number z is taken in $[0;2\pi)$. More generally, if $\zeta=\exp(2\mathrm{i}\pi k/d)$ is a primitive d-root of unity, one can define a function C_ζ by replacing d by d/k and 1.2 should hold for dth roots of $(\boldsymbol{\pi}/\boldsymbol{\pi}_I)^k$. Note also that Craven's function is additive: it satisfies $C_\zeta(PQ)=C_\zeta(P)+C_\zeta(Q)$.

For groups of type A, the degree $\deg\chi_\lambda$ of the unipotent character χ_λ is explicitely known (see for example [5, Section 13]). It is a polynomial in q of degree A_λ and valuation a_λ and no factors of the form $(q-1)$ can appear. In particular, Craven's function can be written

$$C_\zeta(\deg\chi_\lambda)=\frac{2\pi}{\mathrm{Arg}\,\zeta}\big(a_\lambda+A_\lambda\big)+\#\big\{i=1,\ldots,m\,|\,\mathrm{Arg}\,z_i<\mathrm{Arg}\,\zeta\big\}$$

where $z_1,\ldots,z_m$ are the roots with multiplicity of the polynomial $\deg\chi_\lambda$. Note that with this description it is already not obvious that the rational number on the right-hand side of 1.2 is actually an integer.

Since C_d is additive, formula 1.2 together with the quasi-isomorphism 1.1 suggests that the partition ν should not be necessarily a d-core. In the case of an elementary d-induction (when $a=1$) we can write everything explicitely using [7, Proposition 9.1]; the second equality follows from an easy calculation:

Lemma 1.3. *Let μ be a partition with corresponding β-set X that we assume to be large enough. For $x\in X'$, we have*

$$C_d(\deg\chi_{\mu*x})-C_d(\deg\chi_\mu)\;=\;2\big(n+1-d-x+\#\{y\in X\,|\,y<x\}\big)+\#\{y\in X\,|\,x<y<x+d\}$$

and
$$a_{\mu*x}+A_{\mu*x}-a_\mu-A_\mu\;=\;d(n-d+s-x).$$

These integers give conjecturally the degree of the cohomology group of $\mathrm{X}_{n,d}$ in which $\chi_{\mu*x}$ will appear, as well as the corresponding eigenvalue of the Frobenius. Since we will work with the cohomology with compact support, we shall rather work with the integers

$$\pi_d(X,x) = 2\big(n + x - \#\{y \in X \mid y < x\}\big) - \#\{y \in X \mid x < y < x + d\}$$

and
$$\gamma_d(X,x) = n + 1 + x - s.$$

They are readily deduced from the previous ones by taking into account the dimension of $\mathrm{X}_{n,d}$ (which is equal to $\ell(v_d) = 2n+1-d$). Now Conjecture 1 can be deduced from the following:

Conjecture 1.4. *Let $n \geq 1$ be a positive integer and $1 \leq d \leq n+1$. Let μ be a partition of $n-d+1$ and X be its β-set, assumed to be large enough. Then*

$$\mathrm{R}\Gamma_c\big(\mathrm{X}_{n,d}, \mathcal{F}_\mu\big) \simeq \bigoplus_{x \in X'} V_{\mu * x}[-\pi_d(X,x)] \otimes \overline{\mathbb{Q}}_\ell\big(\gamma_d(X,x)\big)$$

as a complex of $G \times \langle F \rangle$-modules.

The purpose of this paper is to prove that this conjecture holds for any d whenever it holds for $d = 1$. As a byproduct, we shall deduce the cohomology of parabolic Deligne-Lusztig varieties associated to any unipotent block from the knowledge of the cohomology of $\mathrm{X}(\mathbf{w}_0^2)$.

2 Decomposition of the quotient

The group $(\mathbf{L}_{J_d}, \dot{v}_d F)$ is split of type A_{n-d}, therefore the unipotent representations of the corresponding finite group are labelled by partitions μ of $n-d+1$. To such a partition one can associate a unipotent $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}_\mu$ on $\mathrm{X}_{n,d}$. From [3] we know that the irreducible constituents of the virtual character $\sum(-1)^i \mathrm{H}_c^i(\mathrm{X}_{n,d}, \mathcal{F}_\mu)$ correspond to the partitions of $n+1$ obtained by adding a d-hook to μ. The restriction to $\mathfrak{S}_n$ of the corresponding irreducible representation corresponds to a partition obtained by

- either restricting the hook (usually in two different ways),
- or restricting μ.

The main result of this section gives a geometric interpretation of this phenomenon.

Theorem 2.1. *Assume that $d \geq 2$. Let $I = \{s_j \mid 1 \leq j \leq n-1\}$. Let μ be a partition of $n-d+1$ and $\{\mu^{(j)}\}$ be the set of partitions of $n-d$ obtained by restricting μ. Then there is a distinguished triangle in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$*

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathrm{X}_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(\mathrm{X}_{n,d}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c\big(\mathrm{X}_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1) \rightsquigarrow$$

Remark 2.2. From [1, Proposition 1.1] we can deduce that the cohomology of a Deligne-Lusztig variety with coefficients in a unipotent local system depends only on the type of $(\mathbf{G}, F)$. Therefore there is no ambiguity in the statement of the theorem.

We will use the results in [11] to compute the quotient of $\widetilde{\mathrm{X}}(J_d,\dot v_d F)$ by the finite group U_I. Recall that $\mathrm{X}_{n,d} = \mathrm{X}(J_d,v_dF)$ can be decomposed into locally closed P_I-subvarieties X_x, where x is a I-reduced-J_d element of W. In our situation, at most two pieces will appear:

Lemma 2.3. *Assume that $2\le d\le n$. The variety X_x is non-empty if and only if x is one of the two following elements:*

(1) $x_0 = s_n s_{n-1}\cdots s_1$

(2) $x_1 = s_n s_{n-1}\cdots s_{n-\lfloor\frac{d}{2}\rfloor+1}$.

Proof. For simplicity, we shall denote $a=\lfloor\frac{d+1}{2}\rfloor$ and $b=n-\lfloor\frac{d}{2}\rfloor$ so that $w=v_d=s_1\cdots s_b s_n\cdots s_a$ and $J=J_d=\{a+1,\ldots,b\}$. If x is a I-reduced-J element, then $x=s_ns_{n-1}\cdots s_i$ with $b+1\le i\le n+1$ or $1\le i\le a$. Recall from [11] that the variety X_x is non-empty if and only if there exists $y=y_1\cdots y_r\in W_J$ and an x-distinguished subexpression γ of yw such that the products of the elements of γ lies in $(W_I)^x$. We first observe that for $i\notin\{1,n+1\}$ we have

$$(W_I)^x = \langle s_1,\ldots,s_{i-2},s_is_{i-1}s_i,s_{i+1},\ldots,s_n\rangle.$$

Now, since x is reduced-J, the subexpression γ is the concatenation of $(y_1,\ldots,y_r)$ and an xy-distinguished subexpression $\tilde\gamma$ of w. If $i>b+1$ or $2\le i\le a$, the group W_J is included in $(W_I)^x$. Therefore the product of the elements of $\tilde\gamma$ must lie in $(W_I)^x$. We shall distinguish two cases:

Case (1). We assume that $i>b+1$. In that case x commutes with any element of W_J, so that $\tilde\gamma$ is an yx-distinguished subexpression of w. Then

- if x is trivial (that is if $i=n+1$), then any y-distinguished subexpression of w contains necessarily s_n and hence cannot produce any element of $(W_I)^x = W_I$;
- if x is non-trivial then $i\le n$, and a subexpression of w lies in $(W_I)^x$ if and only if it does not contain s_i or s_{i-1}. However, such a subexpression will never be yx-distinguished since for all v in W_I we have $vxs_{i-1}>vx$.

We deduce that the variety X_x is empty in this case.

Case (2). We assume that $2\le i\le a$. The subexpression $\tilde\gamma$ is ${}^xyx$-distinguished. Since $i\le a$, we have ${}^xW_J = W_{a,\ldots,b-1}$. For $j<i-1$, we have $xs_j=s_jx$; moreover, xs_{i-1} is I-reduced, so that $\tilde\gamma$ should start with $(s_1,\ldots,s_{i-1})$. In that case, the product of the elements of $\tilde\gamma$ cannot belong to $(W_I)^x$. Indeed, a subexpression of $s_{i-1}s_i\cdots s_bs_n\cdots s_a$ starting with s_{i-1} will never give an element of $(W_I)^x$, the only non-trivial situation being the case $a=i$:

- with the notation in [10, Section 2.1.2] we have $s_{a-1}\underline{s_{a+1}\cdots s_bs_n\cdots s_a} = \underline{s_{a+1}\cdots s_bs_n\cdots s_{a+1}}s_{a-1}\underline{s_a}$ and neither s_{a-1} nor $s_{a-1}s_a$ belongs to $(W_I)^x$;
- $s_{a-1}s_a\underline{s_{a+1}\cdots s_bs_n\cdots s_a} = (s_{a-1}s_as_{a-1})s_{a-1}\underline{s_{a+1}\cdots s_bs_n\cdots s_a}$ and we are back to the previous case.

This forces the variety X_x to be empty. $\square$

Proof of the Theorem. From the previous lemma we deduce that $\widetilde{\mathrm{X}}(J_d, v_d F)$ decomposes as a disjoint union $\widetilde{\mathrm{X}}(J_d, v_d F) = \widetilde{\mathrm{X}}_{x_0} \cup \widetilde{\mathrm{X}}_{x_1}$ with $\widetilde{\mathrm{X}}_{x_0}$ being open. Using [11] we shall now determine the cohomology of the quotient of each of these varieties by U_I. Throughout the proof, we will denote $\widetilde{I} = \{s_2, \ldots, s_n\} \subset S$ the conjugate of I by w_0.

When $x = x_0 = w_I w_0$ and $d > 2$ we are in the situation of [11, Proposition 3.4]. Indeed, $v_d = s_1 w'$ with $w' \in W_{2,\ldots,n}$ and s_1 commutes with $W_{J_d} \subset W_{3,\ldots,n-1}$ so that we obtain

$$\mathrm{R}\Gamma_c\big(U_I \backslash \widetilde{\mathrm{X}}_{x_0}/N, \overline{\mathbb{Q}}_\ell\big) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \widetilde{\mathrm{X}}_{\mathbf{L}_I}(K_{x_0}, \dot{v}F)/{}^{\dot{x}_0}N', \overline{\mathbb{Q}}_\ell\big)$$

with $v = {}^{x_0}w'$ and $K_{x_0} = I \cap {}^{x_0}\Phi_{J_d} = {}^{x_0}J_d$. For simplicity, we shall rather consider the conjugate by x_0 of the right-hand side

Recall that N and N' are normal subgroup of $\mathbf{L}_{J_d}$ and are both contained in $\mathbf{T}$. In particular, any unipotent character of $\mathbf{L}_{J_d}^{wF}$ (resp. of $\mathbf{L}_{J_d}^{w'F}$) is trivial on N (resp. N'). Consequently, for any unipotent character χ of $\mathbf{L}_{J_d}^{wF}$ we obtain the following quasi-isomorphism of complexes of $L_I \times \langle F \rangle$-modules:

$$\mathrm{R}\Gamma_c\big(U_I \backslash \widetilde{\mathrm{X}}_{x_0}, \overline{\mathbb{Q}}_\ell\big)_\chi \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \widetilde{\mathrm{X}}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F), \overline{\mathbb{Q}}_\ell\big)_{\dot{x}_0\chi}.$$

Finally, we observe that the varieties $\mathrm{X}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F)$ and $\mathrm{X}_{n-1,d-1}$ have the same cohomology with coefficients in any unipotent local system. Indeed, if we denote $(s_1, \ldots, s_{n-1})$ by $(t_1, \ldots, t_{n-1})$ if d is odd or by $(t_{n-1}, \ldots, t_1)$ if d is even, then we have

$$v = t_1 t_2 \cdots t_{n-1-\lfloor \frac{d-1}{2} \rfloor} t_{n-1} \dot{t}_{n-2} \cdots t_{\lfloor \frac{d}{2} \rfloor}$$

which corresponds to the element v_{d-1} in the Weyl group $W_I = \langle t_1, \ldots, t_{n-1} \rangle$ of type A_{n-1}.

When $x = x_0$ and $d = 2$, we can write $v_2 = ww'$ with $w = s_n s_{n-1} \cdots s_2$ and $w' = s_1 s_2 \cdots s_n = s_1 w''$ so that $\mathrm{X}_{n,2} \simeq \mathrm{X}\big(\{\mathbf{s}_2, \ldots, \mathbf{s}_{n-1}\}, \mathbf{w}\mathbf{w}'F\big)$. Moreover, via this isomorphism we have

$$\mathrm{X}_{x_0} \simeq \bigcup_{y \in W} \mathrm{X}_{(x_0, y)}.$$

We claim that $\mathrm{X}_{(x_0,y)}$ is empty unless $y \in W_I w_0 W_{J_2'}$ where $J_2' = J_2^w = \{s_3, s_4, \ldots, s_n\}$. The piece $\mathrm{X}_{(x_0,y)}$ consists of pairs $(p x_0 \mathbf{P}_{J_2}, p' y \mathbf{P}_{J_2'})$ with $p, p' \in \mathbf{P}_I$ such that $p^{-1}p' \in x_0 \mathbf{P}_{J_2} w \mathbf{P}_{J_2'} y^{-1}$ and $p'^{-1}\,{}^F p \in y \mathbf{P}_{J_2'} w' \mathbf{P}_{J_2} x_0^{-1}$. In particular, if $\mathrm{X}_{(x_0,y)}$ is non-empty then the double coset $\mathbf{P}_I y \mathbf{P}_{J_2'}$ has a non-trivial intersection with $x_0 \mathbf{B} w$. But $x_0 = w_I w_0$ is reduced-$\widetilde{I}$ so that $\ell(x_0 w) = \ell(x_0) + \ell(w)$ and $x_0 \mathbf{B} w \subset \mathbf{P}_I w_0 \mathbf{P}_{J_2'}$. This forces y to lie in $W_I w_0 W_{J_2'}$. Note that $w_0(J_2') \subset I$ so that the minimal element in this coset is x_0 and we have $\mathrm{X}_{x_0} \simeq \mathrm{X}_{(x_0,x_0)}$.

Now s_1 commutes with J_2' and we can apply [11, Proposition 3.4] to obtain, after conjugation by x_0:

$$\mathrm{R}\Gamma_c\big(U_I \backslash \widetilde{\mathrm{X}}_{x_0}/N, \overline{\mathbb{Q}}_\ell\big) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \widetilde{\mathrm{X}}_{\mathbf{L}_I}(\mathbf{K}_{x_0}, \mathbf{v}\mathbf{v}'F)/{}^{\dot{x}_0}N', \overline{\mathbb{Q}}_\ell\big).$$

with $K_{x_0} = {}^{x_0}J_2$, $v = {}^{x_0}w$ and $v' = {}^{x_0}w''$. If we denote $(s_1,\ldots,s_{n-1})$ by $(t_{n-1},\ldots,t_1)$ we obtain ${}^{x_0}J_2 = \{t_2,\ldots,t_{n-1}\}$ and $\mathbf{v}\mathbf{v}' = \mathbf{t}_1\cdots\mathbf{t}_{n-1}\mathbf{t}_{n-1}\cdots\mathbf{t}_1$ so that the pair $(\mathbf{K}_{x_0},\mathbf{v}\mathbf{v}')$ corresponds to $(\mathbf{J}_1,\mathbf{v}_1)$ in the Weyl group W_I of type A_{n-1}. As before, N and N' do not play any role if we consider the unipotent part of the previous quasi-isomorphism.

When $x = x_1$ we use [11, Proposition 3.2]: the conjugate of v_d by x_1 is

$$v = x_1 v_d x_1^{-1} = s_1 s_2 \cdots s_{n-1-\lfloor \frac{d}{2}\rfloor} s_{n-1} s_{n-2}\cdots s_{\lfloor\frac{d+1}{2}\rfloor}$$

which corresponds exactly to the element v_d in W_I. We can therefore identify the cohomology of the varieties $\mathrm{X}_{\mathbf{L}_I}(K_{x_1}, vF)$ and $\mathrm{X}_{n-1,d}$ with coefficients in any unipotent local system (see Remark 2.2). The group $\mathbf{P}_I \cap {}^{x_1}\mathbf{L}_{J_d}$ is a $\dot{v}F$-stable parabolic subgroup of ${}^{x_1}\mathbf{L}_{J_d}$ and $\mathbf{L}_{K_{x_1}} = \mathbf{L}_I \cap {}^{x_1}\mathbf{L}_{J_d}$ is a stable Levi complement. Therefore it makes sense to consider the Harish-Chandra restriction ${}^*\mathrm{R}_K^J\chi$ of any unipotent character χ of $\mathbf{L}_{J_d}^{\dot{v}_dF} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$. From [11, Proposition 3.2] (see also [11, Remark 3.12]) we deduce the following quasi-isomorphism

$$\mathrm{R}\Gamma_c\big(U_I\backslash\widetilde{\mathrm{X}}_{x_1},\overline{\mathbb{Q}}_\ell\big)_\chi[2](-1) \simeq \mathrm{R}\Gamma_c\big(\widetilde{\mathrm{X}}_{\mathbf{L}_I}(K_{x_1},\dot{v}F),\overline{\mathbb{Q}}_\ell\big)_{{}^*\mathrm{R}_K^J\chi}.$$

Let μ be a partition of $n-d+1$. The cohomology of the variety $\mathrm{X}_{n,d}$ with coefficients in the local system $\mathcal{F}_\mu$ is given by

$$\mathrm{R}\Gamma_c(\mathrm{X}_{n,d},\mathcal{F}_\mu) \simeq \mathrm{R}\Gamma_c\big(\widetilde{\mathrm{X}}(\mathbf{J}_d,\mathbf{v}_dF),\overline{\mathbb{Q}}_\ell\big)_{\chi_\mu}$$

where χ_μ is the unipotent character of $\mathbf{L}_{J_d}^{\dot{v}_dF}$ corresponding to the partition μ. Since $(\mathbf{L}_{K_{x_1}},\dot{v}F)$ is a split group of type A_{n-d-1}, the Harish-Chandra restriction of χ_μ from $\mathbf{L}_{J_d}^{\dot{w}F} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$ is the sum of the χ_{μ_i}'s where the μ_i's are the partitions of $n-d$ obtained by restricting μ. With with description, we get the following isomorphisms in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F\rangle\text{-mod})$

$$\mathrm{R}\Gamma_c\big(U_I\backslash\widetilde{\mathrm{X}}_{x_0},\overline{\mathbb{Q}}_\ell\big)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m\times\mathrm{X}_{n-1,d-1},\overline{\mathbb{Q}}_\ell\otimes\mathcal{F}_\mu\big)$$

and $$\mathrm{R}\Gamma_c\big(U_I\backslash\widetilde{\mathrm{X}}_{x_1},\overline{\mathbb{Q}}_\ell\big)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c\big(\mathrm{X}_{n-1,d},\bigoplus\mathcal{F}_{\mu_i}\big)[-2](1).$$

We conclude using the distinguished triangle associated to the decomposition $\widetilde{\mathrm{X}}_{n,d} = \widetilde{\mathrm{X}}_{x_0}\cup\widetilde{\mathrm{X}}_{x_1}$ in which $\widetilde{\mathrm{X}}_{x_0}$ is open. □

3 Cohomology over $\overline{\mathbb{Q}}_\ell$

We have just seen how to relate the Harish-Chandra restriction of the cohomology of $\mathrm{X}_{n,d}$ to the cohomology of smaller parabolic Deligne-Lusztig varieties. We shall now explain how this strategy provides an inductive method for a thorough determination of the cohomology of $\mathrm{X}_{n,d}$ with coefficients in any unipotent local system. The main result in this section gives an inductive strategy towards a proof of Conjecture 1:

Theorem 3.1. *Let $n \geq 1$ and $2 \leq d \leq n$. If Conjecture 1.4 holds for $(n,d+1)$, $(n-1,d-1)$ and $(n-1,d)$ then it holds for (n,d).*

Note that we already know from [12] that Conjecture 1.4 holds in the Coxeter case, corresponding to $(n,n+1)$. Therefore $d=1$ is the only limit case. But $\boldsymbol{\pi} = \mathbf{w}_0^2$ is a maximal 1-periodic element in the sense of [9] and in that specific case, a general conjecture for the cohomology has been already formulated in [4]: a unipotent character χ_λ can appear in $\mathrm{H}_c^i(\mathrm{X}(\boldsymbol{\pi}))$ for $i = 4\nu_{\mathbf{G}} - 2A_\lambda$ only, where $\nu_{\mathbf{G}}$ is the number of positive roots. An important consequence of Theorem 3.1 is that knowing the cohomology of $\mathrm{X}(\boldsymbol{\pi})$ is sufficient for determining all the other interesting cases:

Corollary 3.2. *For groups of type A, Conjecture 1 holds for any $d \geq 1$ as soon as it holds for $d=1$.*

Proof. Assume that Conjecture 1 holds for $d=1$, that is for the variety $\mathrm{X}(\boldsymbol{\pi})$. Let $I = J_1 = \{s_2,\ldots,s_n\}$ and $\mathbf{b} = \mathbf{v}_1 = \mathbf{s}_1 \cdots \mathbf{s}_n \mathbf{s}_n \cdots \mathbf{s}_1$. By [9, Proposition 8.26] we have

$$\mathrm{R}\Gamma_c\big(\mathrm{X}(\boldsymbol{\pi}),\overline{\mathbb{Q}}_\ell\big) \simeq \mathrm{R}\Gamma_c\big(\widetilde{\mathrm{X}}(\mathbf{I},\mathbf{b}F),\overline{\mathbb{Q}}_\ell\big) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}_I^{t(\mathbf{b})F}} \mathrm{R}\Gamma_c\big(\mathrm{X}(\boldsymbol{\pi}_I),\overline{\mathbb{Q}}_\ell\big).$$

Since the cohomology of $\mathrm{X}(\boldsymbol{\pi}_I)$ contains all the unipotent characters of $\mathbf{L}_I^{t(\mathbf{b})F}$, we deduce that for any partition μ of n, the groups $\mathrm{H}_c^i(\mathrm{X}_{n,1},\mathcal{F}_\mu)$ are submodules of the cohomology groups of $\mathrm{X}(\boldsymbol{\pi})$. Consequently, they are disjoint as soon as Conjecture 1 holds for $\mathrm{X}(\boldsymbol{\pi})$. Since we have assumed that it holds also for $\mathrm{X}(\boldsymbol{\pi}_I)$ we have actually

$$\mathrm{H}_c^i\big(\mathrm{X}(\boldsymbol{\pi}),\overline{\mathbb{Q}}_\ell\big) \simeq \bigoplus_{\mu - n} \mathrm{H}_c^{i-4\nu_{\mathbf{L}_I}+2A_\mu}(\mathrm{X}_{n,1},\mathcal{F}_\mu)(2\nu_{\mathbf{L}_I} - a_\mu - A_\mu) \tag{3.3}$$

as a $G \times \langle F\rangle$-module. Now, the alternating sum of the cohomology groups of $\mathrm{X}_{n,1}$ represents the Deligne-Lusztig induction from $\mathbf{L}_I^{t(\mathbf{b})F} \simeq L_I$ to G. Therefore a character χ_λ appear in $\mathrm{H}_c^\bullet(\mathrm{X}_{n,1},\mathcal{F}_\mu)$ if and only if μ is the restriction of λ, or equivalently, if λ is obtained from μ by adding a 1-hook. This, together with 3.3 and Lemma 1.3 proves that Conjecture 1.4 holds for $\mathrm{X}_{n,1}$, and therefore for any variety $\mathrm{X}_{n,d}$ by 3.1. We use 1.1 to conclude. ☐

Proof of the Theorem. Let X be a β-set associated the partition μ of $n-d+1$. We can always assume that it contains $\{0,1,\ldots,d-1\}$. The partitions $\mu^{(j)}$'s of $n-d$ which are obtained by restricting μ can be associated to the following β-set:

$$X^{(j)} = \{x_1^{(j)} < \cdots < x_s^{(j)}\} \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_j - 1 & \text{if } i = j; \\ x_i & \text{otherwise.} \end{cases}$$

Let $I = \{s_1,\ldots,s_{n-1}\}$. By Theorem 2.1, the Harish-Chandra restriction of the cohomology of $\mathrm{X}_{n,d}$ can be fitted into the following distinguished triangle:

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathrm{X}_{n-1,d-1},\overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(\mathrm{X}_{n,d},\mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c\big(\mathrm{X}_{n-1,d},\bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1) \rightsquigarrow$$

Now, if we assume that Conjecture 1.4 holds for both $(n-1,d-1)$ and $(n-1,d)$, the complexes on the left and right-hand side are completely determined. Let us examine the different eigenvalues of F that can appear:

(a) on $\mathscr{C} = \mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathrm{X}_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$, the eigenvalues of F are q^{n+x-s} and $q^{n+1+x-s}$ with $x \in X$ such that $x+d-1 \notin X$. The character of the corresponding eigenspace is χ_λ where λ is the partition of n obtained by adding to μ a $(d-1)$-hook represented by x;

(b) on $\mathscr{D}^{(j)} = \mathrm{R}\Gamma_c(\mathrm{X}_{n-1,d}, \mathcal{F}_{\mu^{(j)}})[-2](1)$, the eigenvalues of F are $q^{n+1+x-s}$ where $x \in X^{(j)}$ is such that $x+d \notin X^{(j)}$. The character of the corresponding eigenspace is χ_λ where λ is the partition of n obtained by adding to $\mu^{(j)}$ a d-hook represented by x.

We shall now determine $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d}, \mathcal{F}_\mu)^{U_I}$ by studying each eigenspace of F separately. For x a positive integer, we can separate the following cases:

Case (1). Assume first that $x \in X$ and $x+d \notin X$. Let $\lambda = \mu * x$ be the partition of $n+1$ obtained by adding to μ a d-hook from x. We want to prove that the $q^{n+1-s+x}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d}, \mathcal{F}_\mu)^{U_I}$ is non-zero in degree $\pi_d(X,x)$ only and that its character is the Harish-Chandra restriction of χ_λ.

By (a), the $q^{n+1+x-s}$-eigenspace of F on $\mathscr{C}$ will produce non-zero representations in the following two cases:

- if $x+d-1 \notin X$, then one obtains a character associated to the β-set $(X \smallsetminus \{x\}) \cup \{x+d-1\}$ and it is concentrated in degree

$$\begin{aligned} 2+\pi_{d-1}(X,x) &= 2+2\big(n-1+x-\#\{y \in X \mid y<x\}\big) - \#\{y \in X \mid x<y<x+d-1\} \\ &= 2\big(n+x-\#\{y \in X \mid y<x\}\big) - \#\{y \in X \mid x<y<x+d-1\} \\ &= \pi_d(X,x). \end{aligned}$$

- if $x+1 \in X$, then the corresponding β-set is $(X \smallsetminus \{x+1\}) \cup \{x+d\}$ and the associated character appears in degree $1+\pi_{d-1}(X,x+1)$ only. But we have

$$\begin{aligned} \pi_{d-1}(X,x+1) &= 2\big(n+x-\#\{y \in X \mid y<x+1\}\big) - \#\{y \in X \mid x+1<y<x+d\} \\ &= 2\big(n+x-1-\#\{y \in X \mid y<x\}\big) - \big(\#\{y \in X \mid x<y<x+d\}-1\big) \\ &= \pi_d(X,x)-1. \end{aligned}$$

On the other hand, the $q^{n+1+x-s}$-eigenspace of F on $\mathscr{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x+d \notin X^{(j)}$. This happens if and only if x and $x+d+1$ are different from x_j. In that case, the β-set corresponding to the character of the eigenspace will be $(X^{(j)} \smallsetminus \{x\}) \cup \{x+d\}$. Furthermore, the degree in which this character will appear is $2+\pi_d(X^{(j)},x)$, which is clearly equal to $\pi_d(X,x)$ in that case.

Now, the β-set $Y = (X \smallsetminus \{x\}) \cup \{x+d\}$ is associated to the partition $\lambda = \mu * x$. As mentioned in the beginning of Section 2, the restriction of λ is obtained by restricting the hook (usually in two different ways) or by restricting μ. In the framework of β-sets, it corresponds to decreasing specific elements of Y:

- if $x+d-1 \notin X$, one can replace $x+d$ by $x+d-1$ in Y and we obtain the β-set $(X \smallsetminus \{x\}) \cup \{x+d-1\}$;

- if $x+1 \in X$, one can replace $x+1$ by x in Y and we obtain the β-set $(X \smallsetminus \{x+1\}) \cup \{x+d\}$;
- if $x_j \in X$ is different from x or from $x+d+1$, and if $x_j - 1 \notin X$, then one can replace x_j by $x_j - 1$ in Y to obtain the β-set $(X^{(j)} \smallsetminus \{x\}) \cup \{x+d\}$.

This shows that the character of the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d}, \mathcal{F}_\mu)^{U_I}$ is the Harish-Chandra restriction of $\chi_{\mu * x}$.

Lemma 3.4. *Let λ be a partition of $n+1$, with $n \geq 3$ and let χ be a (non-necessarily irreducible) unipotent character of G. Then the Harish-Chandra restriction of χ and χ_λ are equal if and only if $\chi = \chi_\lambda$.*

Proof of the Lemma. Assume that there exists a partition $\nu = \{\nu_1 \leq \nu_2 \leq \cdots \leq \nu_r\}$ of $n+1$ with $\nu_1 \neq 0$ such that the difference between the Harish-Chandra restriction of χ_λ and χ_ν is still a unipotent character. This means that in the Young diagram of ν, any box that can be removed can be replaced to form the Young diagram of λ. If $\nu \neq \lambda$, this is possible only if $\nu_1 = \nu_2 = \cdots = \nu_r$.

Let χ be a unipotent character of G which has the same Harish-Chandra restriction as χ_λ. If $\chi \neq \chi_\lambda$, we deduce from the previous argument that all the irreducible constituents of χ are of the form χ_ν with $\nu = (a, a, \ldots, a)$. This can happen if and only if $n = 2$ and $\lambda = (1,2)$. ☐

When $n \geq 3$, we deduce that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d}, \mathcal{F}_\mu)$ is actually $\chi_{\mu * x}$. If $n = 2$, then the only ambiguity concerns $\chi_{\mu * x}$ when $\mu * x = (1,2)$. In that case, the $q^{n+1+x-s}$-eigenspace can be either $\chi_{\mu * x}$ or $1_G + \mathrm{St}_G$. But by [9, Corollary 8.28], the trivial character and the Steinberg character cannot occur in the same cohomology group of $\mathrm{X}_{n,d}$ as soon as the dimension of this variety is non-zero.

Case (2). Assume now that $x \notin X$. The $q^{n+1+x-s}$-eigenspace of F on $\mathscr{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. Since $x \notin X$, this forces $x = x_j^{(j)} = x_j - 1$ and $x_j + d - 1 \notin X$. In that case, its character corresponds to a partition with β-set $(X \smallsetminus \{x+1\}) \cup \{x+d\}$ and it appears in degree $2 + \pi_d(X^{(j)}, x)$ only. On the other hand, the $q^{n+1+x-s}$-eigenspace of F on $\mathscr{C}$ is non-zero if and only if $x+1 \in X$ and $x+1+d-1 = x+d \notin X$. By (a), the character of this eigenspace corresponds to a partition with β-set $(X \smallsetminus \{x+1\}) \cup \{x+d\}$. Furthermore, it appears in degree $1 + \pi_{d-1}(X, x+1)$ only. Note that in that case we have

$$\begin{aligned} \pi_{d-1}(X, x+1) &= 2\big(n + x - \#\{y \in X \mid y < x+1\}\big) - \#\{y \in X \mid x+1 < y < x+d\} \\ &= 2\big(n + x - \#\{y \in X \mid y < x\}\big) - \big(\#\{y \in X \mid x < y < x+d\} - 1\big) \\ &= \pi_d(X, x) + 1 \end{aligned}$$

and $$\begin{aligned} \pi_d(X^{(j)}, x) &= 2\big(n - 1 + x - \#\{y \in X^{(j)} \mid y < x\}\big) - \#\{y \in X^{(j)} \mid x < y < x+d\} \\ &= 2\big(n - 1 + x - \#\{y \in X \mid y < x\}\big) - \big(\#\{y \in X \mid x < y < x+d\} - 1\big) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

We deduce that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)^{U_I}$ is isotypic and concentrated in two consecutive degrees. However, there are only a few unipotent characters that can have an isotypic Harish-Chandra restriction: they correspond to partitions of the form $(a,a,\ldots,a)$. Among them we can find the Steinberg character St_G (with $a=1$) and the trivial character 1_G (with $a=n+1$). But by [9, Corollary 8.28] they have respective eigenvalues 1 and q^{2n+1-d}. Let us write the β-set of μ as $X=\{0,1,\ldots,k-1,\mu_1+k,\mu_2+k+1,\ldots,\mu_r+s-1\}$ with $k\geq d$. Since $x\notin X$, one must have $k-1<x<\mu_r+s-1$ and hence

$$d-1\leq n+k-s<n+1+x-s<n+1+\mu_r-1\leq 2n+1-d. \tag{3.5}$$

We deduce that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)$ is either zero, or consists of two copies of the character χ_λ in two consecutive degrees, where $\lambda=(a,a,\ldots,a)$ with $1<a<n+1$. We shall actually prove that it is always zero, but before that we need to study the last case.

Remark 3.6. The Harish-Chandra restriction of χ_λ corresponds to the partition $(a-1,a,\ldots,a)$. Therefore if the associated character appears in the cohomology of $\mathscr{C}$ then the β-set $(X\smallsetminus\{x+1\})\cup\{x+d\}$ must correspond to the partition $(a-1,a,\ldots,a)$. This gives a rather strong condition on X: we will have either

$$X=\{0,1,\ldots,k-1,x+1,b,b+2,b+3,\ldots,\widehat{x+d},\ldots,b+r\}$$

with $b+2\leq x+d\leq b+r$, or

$$X=\{0,1,\ldots,k-1,x+1,x+d+2,x+d+3,\ldots,x+d+r\}.$$

Case (3). Finally, assume that $x\in X$ and $x+d\in X$. The $q^{n+1+x-s}$-eigenspace of F on $\mathscr{D}^{(j)}$ is non-zero if and only if $x\in X^{(j)}$ and $x+d\notin X^{(j)}$. Since $x+d\in X$, this forces $x+d=x_j$ (and therefore $x_j-1\notin X$). In that case, the character of the eigenspace corresponds to a partition with β-set $(X^{(j)}\smallsetminus\{x\})\cup\{x+d\}=(X\smallsetminus\{x\})\cup\{x+d-1\}$. On $\mathscr{C}$, the Frobenius has a non-zero $q^{n+1+x-s}$-eigenspace if and only if $x+d-1\notin X$ and its character is again associated to the β-set $(X\smallsetminus\{x\})\cup\{x+d-1\}$. This ensures that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)^{U_I}$ is isotypic. Using $x+d-1$ instead of x in the inequalities 3.5 yields $0<n+1+x-s<2n+2-2d$ and therefore the previous argument applies. We deduce that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)$ is again either zero or consists of two copies of the character χ_λ in two consecutive degrees, namely $\pi_d(X,x)-1$ and $\pi_d(X,x)$, where $\lambda=(a,a,\ldots,a)$ and $1<a<n+1$.

To conclude, we need to prove that the $q^{n+1+x-s}$-eigenspaces of F are actually zero whenever $x\notin X$ or $x+d\in X$. Let us first summarize what we have proven so far:

(1) if $x\in X$ and $x+d\notin X$ then the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)$ is $\chi_{\mu*x}$ and it appears in degree $\pi_d(X,x)$ only;

(2) if $x\notin X$, the $q^{n+1+x-s}$-eigenspace of F is zero unless $x+1\in X$ and $x+d\notin X$. In that case, it may consist of two copies of χ_λ, one in degree $\pi_d(X,x)+1$ and one in degree $\pi_d(X,x)+2$, where $\lambda=(a,a,\ldots,a)$ with $1<a<n+1$.

Moreover, the β-set $(X \smallsetminus \{x+1\}) \cup \{x+d\}$ must correspond to the partition $(a-1,a,\ldots,a)$ (see Remark 3.6);

(3) if $x \in X$ and $x+d \in X$, the $q^{n+1+x-s}$-eigenspace of F is zero unless $x+d-1 \notin X$. In that case, it can only be χ_λ-isotypic with $\lambda = (a,a,\ldots,a)$ and $1 < a < n+1$. Moreover, it is non-zero in degrees $\pi_d(X,x)-1$ and $\pi_d(X,x)$ only, and $(X \smallsetminus \{x\}) \cup \{x+d-1\}$ must be a β-set of the partition $(a-1,a,\ldots,a)$.

Now, if we assume that Conjecture 1.4 holds for the variety $\mathrm{X}_{n,d+1}$, then we can use the distinguished triangle

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathrm{X}_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(\mathrm{X}_{n+1,d+1}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c\big(\mathrm{X}_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1) \rightsquigarrow$$

from Theorem 2.1 to prove that the eigenspaces of F on $\mathrm{H}_c^\bullet(\mathrm{X}_{n,d},\mathcal{F}_\mu)$ in cases (2) and (3) are indeed zero.

Assume that $x \notin X$ and that there exists $1 < a < n+1$ such that the character $\chi_\lambda = \chi_{(a,a,\ldots,a)}$ appears twice in the $q^{n+1+x-s}$-eigenspace of F on the cohomology of $\mathrm{X}_{n,d}$ – that is in degrees $\pi_d(X,x)+1$ and $\pi_d(X,x)+2$. Then,

- if $x-1 \notin X$, the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathbb{G}_m \times \mathrm{X}_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ is χ_λ-isotypic by (2) (we have $x-1+1 \notin X$). Moreover, the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet\big(\mathrm{X}_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1)$ is zero since Conjecture 1.4 holds for the variety $\mathrm{X}_{n,d+1}$. We deduce that the eigenspace on $\mathrm{H}_c^\bullet(\mathrm{X}_{n+1,d+1},\mathcal{F}_\mu)^{U_I}$ is χ_λ-isotypic, which is impossible since no unipotent character can have χ_λ as a Harish-Chandra restriction when $1 < a < n+1$.
- if $x-1 \in X$, then since $x-1+d+1 \notin X$, the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet(\mathbb{G}_m \times \mathrm{X}_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ and $\mathrm{H}_c^\bullet\big(\mathrm{X}_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1)$ can be determined as in case (1). It corresponds to the Harish-Chandra restriction of the partition $\mu * (x-1)$ obtained from μ by adding a $(d+1)$-hook from $x-1$. Furthermore, they will appear in degree $\pi_{d+1}(X,x-1)$ only, which is equal to

$$\begin{aligned}\pi_{d+1}(X,x-1) &= 2\big(n+x-\#\{y\in X \mid y<x-1\}\big) - \#\{y \in X \mid x-1<y<x+d\} \\ &= 2\big(n+x+1-\#\{y\in X \mid y<x\}\big) - \#\{y \in X \mid x<y<x+d\} \\ &= \pi_d(X,x)+2.\end{aligned}$$

  To these characters we have to add the contribution of χ_λ and possibly of an other character $\chi_{\lambda'}$ corresponding to $\lambda = (a',a',\ldots,a')$ (when the case (3) applies to $x-1$). Now, we claim that neither χ_λ nor χ'_λ can appear in the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^\bullet\big(\mathrm{X}_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}}\big)[-2](1)$. Indeed the assumptions on x force X (see Remark 3.6) to be either

$$X = \{0,1,\ldots,k-1,k+1,b,b+2,b+3,\ldots,\widehat{k+d},\ldots,b+r\}$$

  with $b+2 \le k+d \le b+r$, or

$$X = \{0,1,\ldots,k-1,k+1,k+d+2,k+d+3,\ldots,k+d+r\}.$$

  Therefore a β-set corresponding to the partition $\mu*(x-1)$ of $n+2$ is either

$$\{0,1,\ldots,k-2,k+1,b,b+2,b+3,\ldots,b+r\}$$

with $b+2 \le k+d \le b+r$, or

$$\{0,1,\ldots,k-2,k+1,k+d,k+d+2,k+d+3,\ldots,k+d+r\}.$$

We deduce that the restriction of $\mu*(x-1)$ will never produce λ or λ' unless $r=2$, $b=k+2$ and $d=4$ in the first case, or $r=2$ and $d=2$ in the second case. In these very specific cases, we have either $X=\{0,\ldots,k-1,k+1,k+2\}$, which corresponds to the partition $\mu=(1,1)$ or $X=\{0,\ldots,k-1,k+1,k+4\}$, which corresponds to $\mu=(1,3)$. In this situation, we get $\lambda=(3,3)$ and $\mu*(x-1)=(2,2,3)$. But $(3,3)$ cannot be obtain by restricting $\mu*(x-1)$. This proves that the $q^{n+1+x-s}$-eigenspace of F on $\mathrm{H}_c^{\pi_d(X,x)+2}(\mathrm{X}_{n+1,d+1},\mathcal{F}_\mu)^{U_I}$ is just χ_λ (plus possibly $\chi_{\lambda'}$), which is impossible by the properties of the Harish-Chandra restriction.

The same argument can be adapted to deal with the case (3), if we rather look at the $q^{n+2+x-s}$-eigenspace and distinguish whether $x+1+d$ is an element of X or not. The details are left to the reader. □

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